Solution to Problem 1) Considering that $\langle x \rangle = \sum_{n=1}^{N} p_n x_n$, and that $x_1 \leq x_2 \leq \cdots \leq x_N$, and also that $p_n \geq 0$, it is readily observed that $\langle x \rangle \geq \sum_{n=1}^{N} p_n x_1 = (\sum_{n=1}^{N} p_n) x_1 = x_1$. Similarly, $\langle x \rangle \leq \sum_{n=1}^{N} p_n x_N = (\sum_{n=1}^{N} p_n) x_N = x_N$. Consequently, $x_1 \leq \langle x \rangle \leq x_N$.
